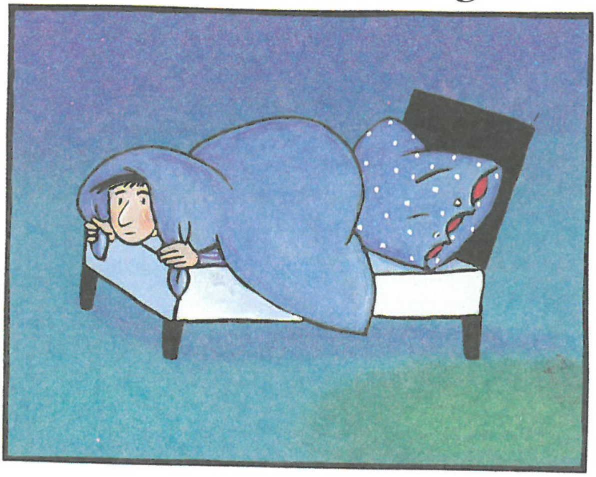
The Tenth Night



Robert was sitting on his backpack in the middle of the snow. He had no idea where he was, but it felt like the North Pole. The cold had crept into his hands and feet, and the snow showed no sign of letting up. It was a real blizzard! No light, no house, no living being as far as the eye could see, and night was coming on. Unless something happened mighty quickly, he was doomed.

Yet even as he tried to warm his stiff, blue hands by clapping them together—he didn't want to freeze to death, after all!—he was aware of another Robert sitting perfectly content in an armchair watching him shudder. So you can dream of yourself dreaming, Robert thought.

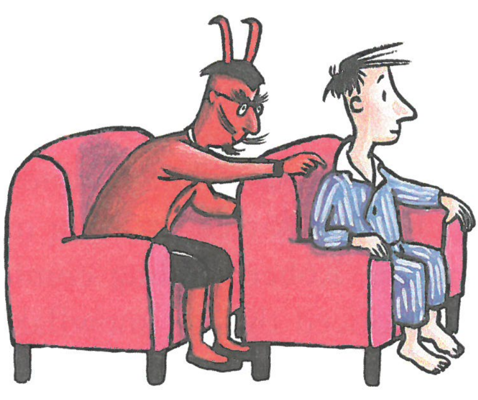
Then the snowflakes swirling around the face of the freezing Robert started growing in size, and the other, warm, Robert, lolling comfortably in his armchair, noticed that no snowflake was like any other. Every one of the large soft flakes was unique. Most had six sides or points, and when Robert looked closer he saw that certain patterns tended to return: hexagonal stars in a hexagonal star, points branching off into smaller and smaller points . . .

Suddenly he felt a hand on his shoulder and heard a familiar voice. "Beautiful, aren't they?" it said.

It was the number devil. He was sitting right behind him.

"Where am I?" Robert asked.

"Just a second," the number devil answered. "I'll turn the light on."



Hexagonal stars in a hexagonal star, points branching off into smaller and smaller points . . . A familiar voice said "Beautiful, aren't they?"

All at once there was a dazzling light, and Robert saw that he was sitting in a small, elegantly appointed auditorium with only two rows of red plush seats.

"A private showing," said the number devil. "Just for you."

"And I thought for sure I was going to freeze to death."

"It was only a film. Here, I have something for you."

It was not another pocket calculator. It was not a sticky green or a furry couch calculator. No, it was a silver-gray beauty, complete with a nifty mouse and a flip-top monitor.

"A computer!"

"Just a little notebook. Only it's rigged up to project everything you input onto the screen in the front of the auditorium. Which means you can draw directly on the screen with your mouse. Shall we begin?"

"Okay, but no more snowstorms, promise? Numbers, no North Pole."

"Bonacci numbers?"

"You and your Bonacci!" exclaimed Robert. "Tell me, is that guy your best friend or something?"

The numbers flashed on the screen as he entered them:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89 ...

"Now try dividing them by their neighbors," the number devil suggested. "The larger by the smaller."

"Okay," said Robert, and he went at it with great gusto, curious about what would come up on the screen:

1 ÷ 1 = 1
2 ÷ 1 = 2
3 ÷ 2 = 1,5
5 ÷ 3 = 1,666666666 ...
8 ÷ 5 = 1,6
13 ÷ 8 = 1,625
21 ÷ 13 = 1,615384615 ...
34 ÷ 21 = 1,619047619 ...
55 ÷ 34 = 1,617647059 ...
89 ÷ 55 = 1,618181818 ...

"Wild!" he said. "Another pile of numbers that refuse to stop. An eighteen biting its own tail. And a few others looking as unreasonable as they come."

"True, true," said the number devil, "but what else do you see?"

Robert thought for a while and said, "All the numbers—they seem kind of wobbly. The second is bigger than the first, the third smaller than the second, the fourth a little bigger, and so on. They keep swaying from side to side. But the farther we go the less they wobble."

"True. The larger the Bonacci numbers, the more you close in on a very special number:

1,618 033 989...

But don't think that's the end of the story, because the number is one of those unreasonable ones, the kind that go on forever. You can take it as far as you like, but you'll never come to the end of it."

"What else would you expect from Bonacci?" said Robert. "But what I don't understand is why they wobble so around that weird number."

"Oh, that," said the number devil. "That's nothing special. They all do that."

"What do you mean—they *all* do that?"

"I mean they don't necessarily need to be Bonaccis. Let's take two perfectly normal, garden-variety numbers. Tell me the first two that come into your head."

"Seventeen and eleven."

"Good. Now add them together."

"That I can do mentally. Twenty-eight."

"Excellent. Now let me show you on the screen where we go from here."

$$11 + 17 = 28$$
$$17 + 28 = 45$$
$$28 + 45 = 73$$
$$45 + 73 = 118$$
$$73 + 118 = 191$$
$$118 + 191 = 309$$

"Got it," said Robert. "What now?"

"We do precisely what we did with the Bonacci numbers. We divide. Go ahead," the number devil said, "and see what you come up with."

Again the numbers flashed on the screen as Robert entered them:

$$17 \div 11 = 1{,}545\ 454 \cdots$$
$$28 \div 17 = 1{,}647\ 058 \cdots$$
$$45 \div 28 = 1{,}607\ 142 \cdots$$
$$73 \div 45 = 1{,}622\ 222 \cdots$$
$$118 \div 73 = 1{,}616\ 438 \cdots$$
$$191 \div 118 = 1{,}618\ 644 \cdots$$
$$309 \div 191 = 1{,}617\ 801 \cdots$$

"That same crazy number!" Robert cried. "What's going on here? Is it buried in all numbers?"

"It is," said the number devil. "And in nature and in art, if you know how to look. By the way—in case you're interested—let me show you what else 1.618 · · · can be."

A monstrous fraction flashed on the screen:

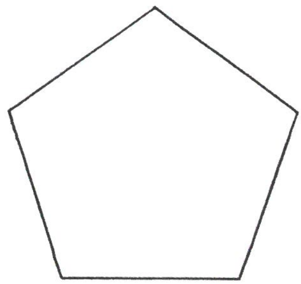
$$1{,}618\cdots = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\cdots}}}}}}$$

"A fraction!" Robert cried. "A fraction that never ever ends, a fraction so grotesque it hurts my eyes to look at it! I hate fractions! Mr. Bockel loves them. He loves to torture us with them. Get that monster out of my sight. Please!"

"Don't panic now. It's just a continued fraction. Though it is amazing that we can entice our 'crazy number' 1.618 · · · out of a set of ones that keep shrinking and shrinking, don't you think?"

"I'll think anything you like so long as you spare me fractions. And most of all, fractions that have no end."

"All right, all right. I just wanted to give you a little surprise. Let me try something else if the continued fraction upsets you so. This pentagon, for instance:



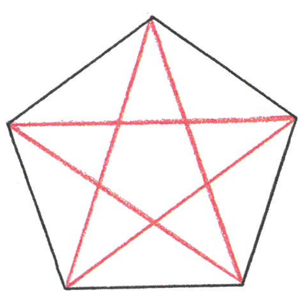
Let's say that each side has the length one."

"One what?" Robert asked immediately. "One meter, one centimeter? Want me to measure it?"

"It doesn't matter," said the number devil, a bit put out. "We had this problem once before, remember? And we agreed to call it one quang. So let's say each side is one quang long. Okay?"

"Okay, okay. Anything you say."

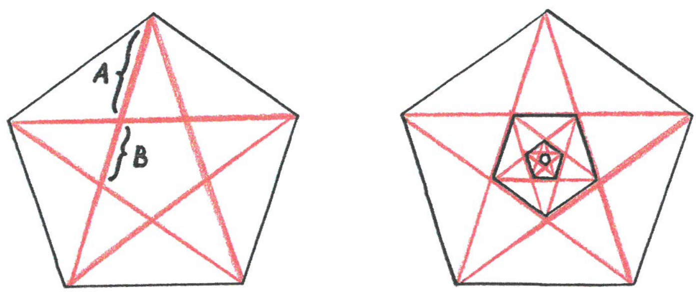
"Now I'm going to draw an orange star inside the pentagon:



The star consists of five orange lines. Ask me how long any one of those lines is and I'll tell you: precisely 1.618 · · · quang and not one iota more or less."

"That is one weird number!"

"You don't know the half of it," said the number devil with a smile, flattered by Robert's interest. "Now pay close attention. I want you to measure the two orange parts I've marked *A* and *B*."



"*A* is slightly longer than *B*," Robert said.

"And just so you don't have to bother your head about it, I can tell you straight away how much longer it is: *A* is precisely 1.618 · · · times longer than *B*. Moreover, we could go on and on—till the cows come home, as you can imagine—because our star is very much like the snowflakes: the orange star has a black pentagon inside it, the black pentagon an orange star inside it, and so on."

"And that blankety-blank unreasonable number keeps turning up?" Robert asked.

"Clearly it does. So if you're not tired of it . . ."

"Not in the least," Robert assured him. "It's fascinating!"

"Then let's go back to your notebook and enter the 'blankety-blank' number. Here, let me dictate:

1,618 033 989...

Good. Now subtract 0.5:

1,618 033 989...-0,5
=1,118 033 989...

Double the result. Times two, in other words:

1,118 033 989...×2
=2,236 067 978...

Good. Now make the new result hop, that is, multiply it by itself. There's a special key for that marked x^2:

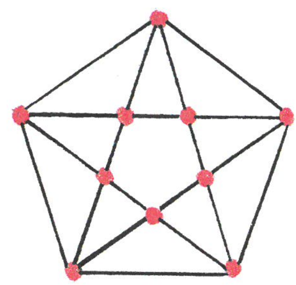
$2,236067977...^2 = 5,000\ 000000$

"Five!" Robert cried. "No! Impossible! How come? Why five?"

"Well," said the number devil with great pleasure, "we have a five-pointed star inside a five-sided figure."

"Devilishly clever of you," said Robert.

"Now let's make a few dots in our star," continued the number devil. "Place one at every point where the lines cross or come together:



Count how many there are."

"Ten," said Robert.

"And now how many white spaces are there?"

Robert counted eleven.

"Now we need to know how many lines there are. Lines connected by two dots."

It took Robert a while to count them because he kept getting mixed up, but he finally got the answer: twenty.

"Correct," said the number devil. "Now look at this:

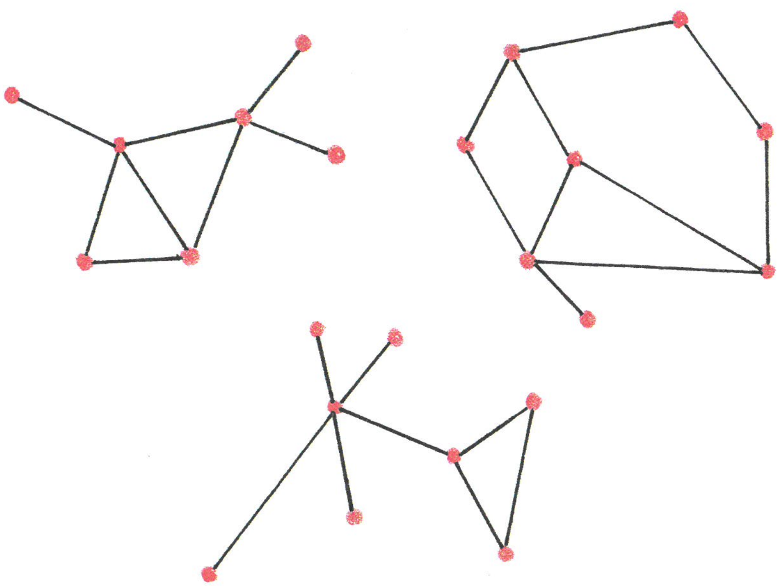
$$10 + 11 - 20 = 1$$
$$(D + S - L = 1)$$

If you add the number of dots and spaces together and subtract the number of lines, the total is one."

"So?"

"The thing is, the total is one not only for our star. No, the total is *always* one, no matter what flat figure you start with. It can be as complicated and irregular as you please. Try it. Draw any old figure and you'll see."

He handed Robert the mouse, and the following figures appeared on the screen:

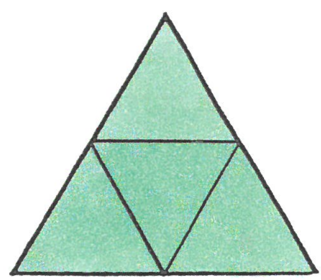


"Don't bother to work it out," said the number devil. "I've done it for you. The first figure has seven dots, two spaces, and eight lines: $7 + 2 - 8 = 1$. The second figure: $8 + 3 - 10 = 1$. The third figure: $8 + 1 - 8 = 1$. Always one.

"By the way, it doesn't work only for flat figures; it works for cubes or pyramids or diamonds as well. The only difference is that then the answer is two rather than one."

"Prove it."

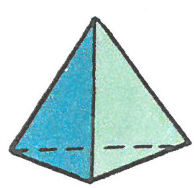
"What you see on the screen is a pyramid."



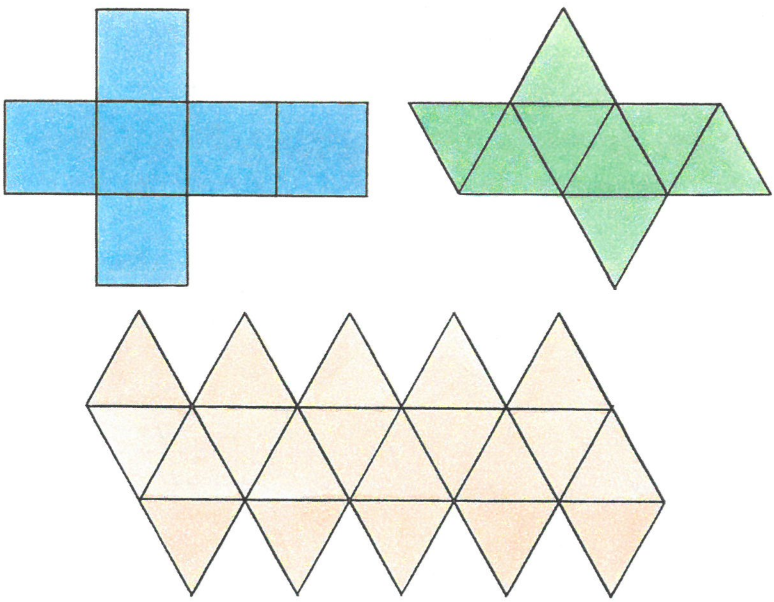
"You call that a pyramid? That's just four triangles."

"But what if you cut and fold it?"

The result flashed immediately on the screen:

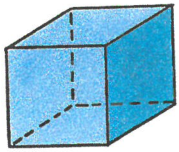
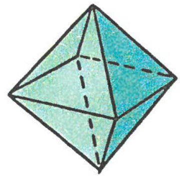
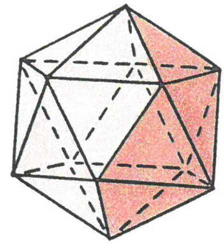


"You can do the same with the following figures," said the number devil, drawing three new shapes on the screen:

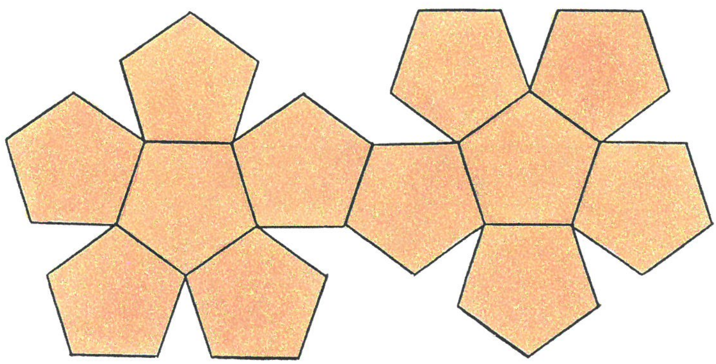


"That's nothing," Robert said. "I've done that before. Cut and paste the first one and you get a cube. But wait a minute, the other two are more complicated."

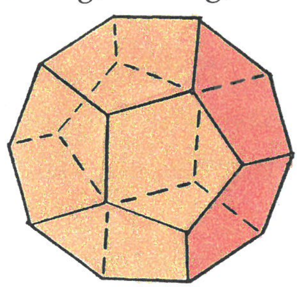
"Let me show you what you'll get. The second will be a double pyramid pointing both up and down, the third an almost spherical object made of twenty equal-sized triangles:

You can even make a kind of ball out of nothing but our favorites, the pentagons. Here's how it looks when you draw it:

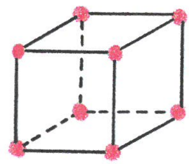


And after the cutting, folding, and pasting:



"Not bad," said Robert. "I think I'll make one."

"Not now, please," said the number devil, “because now I want to get back to our game with the dots, lines, and spaces. Let’s start with a cube. It’s the easiest.”



After counting eight dots, six spaces, and twelve lines, Robert said, “8 + 6 – 12 = 2.”

“Two, as expected,” said the number devil. “No matter what the shape, the result is always two. Dots plus spaces minus lines equals two. No exceptions. Yes, my boy. Anything you can cut and paste. The diamonds on your mother’s ring too. And snowflakes, though they always melt before you’re through counting . . .”

The number devil’s words were quickly growing faint and muffled, the lights in the auditorium fading, and it was beginning to snow on the screen again. This time Robert wasn’t scared, however. He knew where he was and knew he wouldn’t freeze even though things were turning whiter and whiter before his eyes.

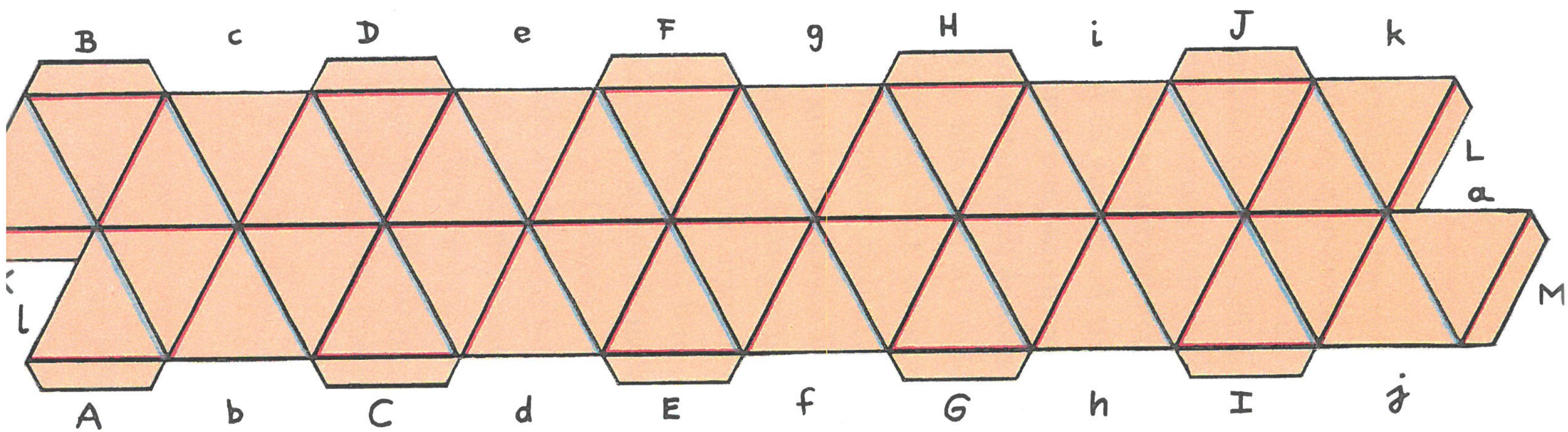
When he woke up, he was lying under a thick white blanket of wool rather than snow, a blanket that had neither dots nor lines nor even what you might call spaces—a blanket that was very definitely four-, not five-sided. And of course the beautiful silver-gray computer had vanished.

What was that blankety-blank number all about, that one point six . . . ? Endless as he knew it was, he could remember no more of it.

Those of you who are handy with scissors and paste may want to try making the figures the number devil showed to Robert. You'll need to draw in little tabs to help you with the pasting. If you do all five and are still game, you can move on to a particularly sophisticated figure—

but only if you are very patient and precise. Take a large sheet of paper (it should be at least 35 × 20 centimeters—that is, 9 × 12 inches—and thick, though cardboard will not work) and copy the figure you see on the next page. Keep in mind that each side of the many triangles must be exactly as long as all the others. You can decide how long you want the sides to be, though three to four centimeters—that is, about an inch and a half (or one quang)—is ideal. Cut out the figure and, using a ruler, fold the paper forward along the red lines and backward along the blue lines. Then paste it together, first the B *tab going with the* b *triangle, the* C *tab with the* c *triangle, etc., and finally the* A *tab with the* a *triangle. What do you get? A crazy ring made of ten little pyramids. You can turn it forward or back (if you're careful!), and each time you do, a new pentagon and a five-pointed star will come into view. Guess what you get if you count up the dots, spaces, and lines and enter them into our equation:*

$$D + S - L = ?$$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