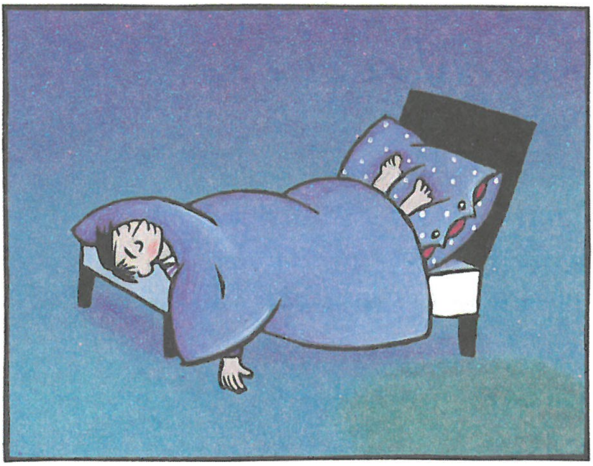
The Eleventh Night



It was nearly dark. Robert was racing though the center of town, though he didn't recognize the streets or buildings. He ran as fast as he could, because Mr. Bockel was after him. Once he was so close that Robert could hear him panting. "Stop!" Mr. Bockel shouted, and Robert spurted forward with all his might. Yet he hadn't the slightest idea why Mr. Bockel was after him or why he was running away from him. Besides, there was no way Mr. Bockel could catch him: the teacher was much too fat.

But when he got to the next corner, what did he see but a second Mr. Bockel tearing out from the left. He bolted across the street, not waiting for the red light to change, and all at once he heard a whole chorus of voices behind him, calling out, "Stop, Robert, stop! We only want to help!"

In addition to the four or five Bockels at his heels and the ones now pouring out of the side streets—

as alike as peas in a pod—there were soon Bockels running straight at him.

Robert shouted for help.

A bony hand grabbed him by the shoulder and pulled him into a doorway. Thank God! It was the number devil.

"Follow me," he whispered. "I know a private elevator that will take us to the top floor."

The elevator was paneled with mirrors, so Robert could see an endless band of number devils and Robert look-alikes.

This is too much, Robert thought, these crowds of the same people!

Still, he was free from the Bockel voices in the street, and when they reached the fiftieth floor and the elevator door opened noiselessly, he and the number devil stepped out into a delightful roof garden.

"You don't know how I've wished for this," Robert said as they rocked peacefully on a swinging bench.

From where they sat, the people down in the street looked like ants.

"I had no idea there were so many Bockels in the world," said Robert.

"Well, they're nothing to you," the number devil said reassuringly. "You needn't be afraid of them."

"I had no idea there were so many Bockels in the world," said Robert.

"I guess it's the kind of thing that happens only in dreams," Robert said. "If you hadn't shown up in time, I wouldn't have known what to think."

"That's why I'm here. And now that there's no one to disturb us, tell me what's wrong."

"All week long I've been brooding over what you showed me last time and how it hangs together. I'm glad you showed me all those tricks. They're fun. But I can't help wondering *why*? Why do they turn out the way they do? The blankety-blank number, for instance. And the five. Why do rabbits behave as if they knew what a Bonacci number is? Why don't unreasonable numbers ever end? And why does what you say hold true always and forever?"

"So that's it," said the number devil. "You want to do more than play around with numbers. You want to know what's behind them. The rules of the game, so to speak. The meaning of it all. In other words, you want to know what a mathematician wants to know."

"I don't know what mathematicians want to know. I only know you've *shown* me things, but never *proved* them."

"You're right," said the number devil. "I apologize. The problem is, showing things is easy and, as you put it, fun. Guessing isn't bad, and testing guesses is even better. We've done a lot of that. But none of it is enough. Proof is all. And now you even want hard proof."

"Right," Robert agreed. "Oh, part of what you say I get just like that. But there are things I just don't get at all. I don't see how or why they work."

"To make a long story short, you're dissatisfied. Well, that's good. Do you think we number devils are always satisfied with what we come up with? Not on your life! No, we're constantly contemplating new ways of proving things. Thinking, pondering, meditating—it's a way of life with us. But when a light finally does go on—and it can take ages, because in mathematics the centuries fly by—then we're pleased as punch. Only then are we satisfied."

"You must be exaggerating. It can't be that hard."

"You have no idea!" the number devil replied. "Even when you think you've understood something, you may wake up one morning and realize there's a catch."

"Can you give me an example?"

The number devil rubbed his chin and paused for a moment.

"You think you know everything there is to know about hopping. What could be simpler than going from 2 to 2×2 and from 2×2 to $2 \times 2 \times 2$?"

"Right: 2^1, 2^2, 2^3, and so on. It's a cinch."

"Yes, but what happens when you hop with zero? 1^0, 8^0, or 100^0? What do you think you get? Shall I tell you? You'll laugh, but the answer is one. Always one:

$$1^0 = 1, \quad 8^0 = 1, \quad 100^0 = 1$$

"How can that be?" Robert was amazed.

"Don't ask. I could prove it to you, but you'd go mad in the process."

"Try me!" said Robert. Now *he* was angry.

But the number devil remained calm.

"Have you ever tried to cross a raging stream?" the number devil asked.

"Have I?" Robert cried. "I'll say I have!"

"You can't swim across: the current would sweep you into the rapids. But there are a few rocks in the middle. So what do you do?"

"I see which ones are close enough together so I can leap from one to the next. If I'm lucky, I make it; if I'm not, I don't."

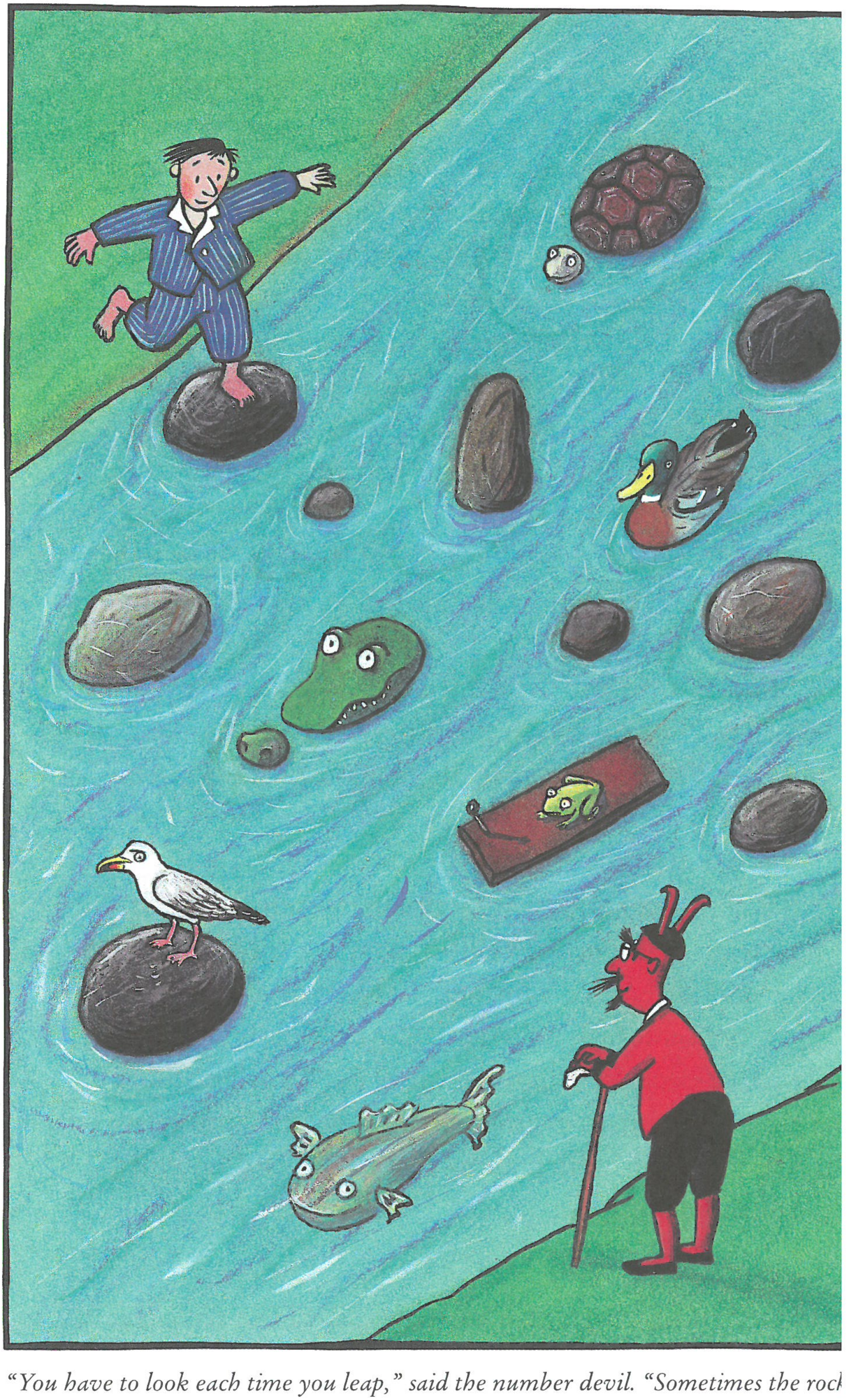
"That's how it is with mathematical proofs," the number devil told Robert. "But since mathematicians have spent a few thousand years finding ways to cross the stream, you don't need to start from scratch. You've got all kinds of rocks to rely on. They've been tested millions of times and are guaranteed slip-resistant. When you have a new idea, a conjecture, you look for the nearest safe rock, and from there you keep leaping—with the greatest of caution, of course—until you reach the other side, the shore."

"That's all well and good," said Robert, "but tell me, where *is* the shore for numbers or pentagons or hopping?"

"Good question," said the number devil. "The shore is a couple of simple sentences. Couldn't be simpler. Once you get to them, you're home free. They're your proof."

"What are they?"

"Well, here's one: Every ordinary number, be it fourteen or fourteen billion, may be followed by one and only one number, namely, that number plus one. Here's another: A point may not be divided, because it has no area. And yet another: Two points on an even plane may be connected by only one line, which then continues endlessly in both directions."



"You have to look each time you leap," said the number devil. "Sometimes the rock

"I see," said Robert. "And starting with a couple of sentences you can leap your way to those blankety-blank numbers or even to the Bonacci numbers?"

"Easily."

The number devil's eyes shone.

"You can go even farther. You just have to look each time you leap. Picture yourself in the middle of the stream. You've got rocks to rely on, remember? Sometimes the rocks are so far apart that you can't make it from one to the next, and if you try jumping, you fall in. Then you have to take tricky detours, and even they may not help in the end. You may come up with an idea, but then you find that it doesn't lead anywhere. Or you may find that your brilliant idea wasn't so brilliant after all."

The number devil looked over at Robert with a gentle smile. "Remember what I showed you the very first time, the numbers I conjured up out of ones?

$$1 \times 1 = 1$$
$$11 \times 11 = 121$$
$$111 \times 111 = 12321$$
$$1111 \times 1111 = 1234321$$

And so on and so forth. It looked as though we could go on indefinitely."

Robert remembered that night well.

"Right, and you got so angry when I tried to say there was something fishy about it. But you know I only said that to get your goat. I had nothing at all to back it up with."

"You had a good nose," the number devil admitted. "Later I went back to it, and you know what? When I came to

$$1\,111\,111\,111 \times 1\,111\,111\,111$$

I fell in! All I got was number hash. So even though the formula worked well enough for a while, it collapsed in the end, without proof.

"In other words, even a number devil can fall on his face. I remember one—the Man in the Moon his name was—who put an idea into a formula he thought would always come out without exception. Well, he tested it a billion five hundred million times—the madman—and each time it worked. He computed himself half to death with his giant computer—he was much, much more precise than we were with our blankety-blank 1.618 · · · —until he was absolutely positive that the formula would work forever. Then he sat back, satisfied.

"It wasn't long, however, before another number devil—I've forgotten his name—came on the scene, and he computed himself three-quarters to death, and with even greater precision, and what did he find?" The number devil looked at Robert. "That the Man in the Moon was wrong. His wonderful formula worked almost always, but not always. And almost isn't enough. The poor devil! He was working on prima-donna numbers. A ticklish business, let me tell you, and fiendishly difficult to prove."

"I agree," said Robert. "Even when only a few lousy pretzels are at stake. It drives me crazy the way Mr. Bockel goes on about why it takes x hours for y bakers to bake pretzels. I much prefer your tricks."

"You're too hard on him. Think of the poor man knocking out lesson plans night after night. He can't go rock leaping as we do whenever we please.

I feel sorry for him. I bet he's gone home to correct your homework."

Robert looked down at the street, and sure enough there wasn't a single Bockel in sight.

"Many of us," the number devil went on, "have an even harder time of it than your Bockel, however. One of my older colleagues, the well-known Lord Rustle, once took it into his head to prove that 1 + 1 = 2. Look at this proof. This is how he went about it."

$*54\cdot42.\ \vdash :: \alpha \in 2 . \supset :. \beta \subset \alpha . \, !\beta . \beta \neq \alpha . \equiv . \beta \in \iota``\alpha$

Dem.

$\vdash . *54\cdot4 . \supset \vdash :: \alpha = \iota`x \cup \iota`y . \supset :.$

$\beta \subset \alpha . \exists ! \beta . \equiv : \beta = \Lambda . \vee . \beta = \iota`x . \vee . \beta = \iota`y . \vee . \beta = \alpha : \exists ! \beta$

$[*24\cdot53\cdot56 . *51\cdot161] \quad \equiv : \beta = \iota`x . \vee . \beta = \iota`y . \vee . \beta = \alpha \quad (1)$

$\vdash . *54\cdot25 . \text{Transp} . *52\cdot22 . \supset \vdash : x \neq y . \supset . \iota`x \cup \iota`y \neq \iota`x . \iota`x \cup \iota`y \neq \iota`y :$

$[*13\cdot12] \supset \vdash : \alpha = \iota`x \cup \iota`y . x \neq y . \supset . \alpha \neq \iota`x . \alpha \neq \iota`y \quad (2)$

$\vdash . (1) . (2) . \supset \vdash :: \alpha = \iota`x \cup \iota`y . x \neq y . \supset :.$

$\beta \subset \alpha . \exists ! \beta . \beta \neq \alpha . \equiv : \beta = \iota`x . \vee . \beta = \iota`y :$

$[*51\cdot235] \quad \equiv : (\exists z) . z \in \alpha . \beta = \iota`z :$

$[*37\cdot6] \quad \equiv : \beta \in \iota``\alpha \quad (3)$

$\vdash . (3) . *11\cdot11\cdot35 . *54\cdot101 . \supset \vdash . \text{Prop}.$

$*54\cdot43.\ \vdash :. \alpha, \beta \in 1 . \supset : \alpha \cap \beta = \Lambda . \equiv . \alpha \cup \beta \in 2$

Dem.

$\vdash . *54\cdot26 . \supset \vdash :. \alpha = \iota`x . \beta = \iota`y . \supset : \alpha \cup \beta \in 2 . \equiv x \neq y .$

$[*51\cdot231] \quad \equiv . \iota`x \cap \iota`y = \Lambda .$

$[*13\cdot12] \quad \equiv . \alpha \cap \beta = \Lambda \quad (1)$

$\vdash . (1) . *11\cdot11\cdot35 . \supset$

$\vdash :. (\exists x, y) . \alpha = \iota`x . \beta = \iota`y . \supset : \alpha \cup \beta \in 2 .$

$\equiv . \alpha \cap \beta = \Lambda \quad (2)$

$\vdash . (2) . *11\cdot54 . *52\cdot1 . \supset \vdash . \text{Prop}.$

"Ugh!" said Robert with a shiver. "That's horrendous. And all to show that $1 + 1 = 2$? Something he knew anyway?"

"That's right." The number devil nodded. "Even though everybody knows that $1 + 1 = 2$, Lord Rustle wanted to prove it. Now you see where that can lead."

The number devil paced up and down, getting carried away.

"By the way," he continued, "there are all sorts of problems that look almost as obvious as $1 + 1 = 2$ and are just as hard to prove mathematically. Let's say you're going to take a few weeks to drive around the country and you have twenty-five friends to visit, each in a different city. You get out your map and look for the shortest route to save the most time and money. What's the best way to go about it?

"Sounds easy, doesn't it? Well, let me tell you, a good many people have lost a good deal of sleep over it. Even some of the most crafty number devils. And not one of them has cracked it, not to this day."

"What's so hard about it?" Robert asked, surprised. "You just go through all of the many possibilities, you chart each of them precisely on the map, and then you work out which is the shortest possible route."

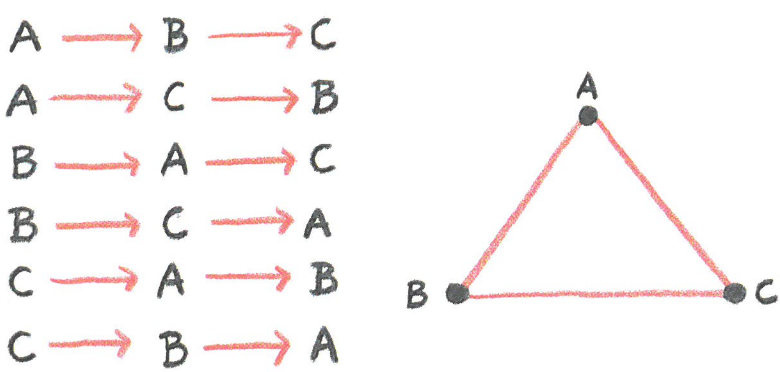
"What you mean is you make a kind of grid with twenty-five dots."

"Right. If I wanted to visit only two friends, there'd be only one route, from *A* to *B:*

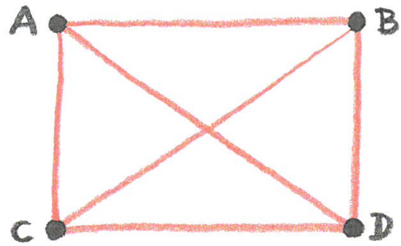
"Well, actually there are two routes. You could also go from *B* to *A.*"

"But that's the same," said Robert. "Now, if I wanted to visit three friends . . ."

"Then there would be six possibilities:



Note that all these routes are equal in length. By the time we have four friends to visit, things are considerably more complex:



"You know," said Robert, "I'm not quite in the mood to count up all the routes."

"Well, there are precisely twenty-four," said the number devil. "It's very much like the seating problem in your class, I'm afraid. You remember the terrible time we had with Al and Betsy and Charlie and the gang because there were so many possibilities?"

"You bet I remember! Three kids, three vroom! Four kids, four vroom! And so on."

"Well, it's the same with the tour problem."

"Then why is it so hard to solve?" Robert asked. "All I need is to find the various routes and pick the shortest."

"That's what you think!" the number devil replied. "With twenty-five friends you have twenty-five vroom! possibilities, and that's an awfully large number. Approximately

1 600 000 000 000 000 000 000 000 0

You can't possibly test them all and determine which is the shortest. It's beyond even the most powerful computer in the world."

"In other words, no way."

"Not necessarily. We've racked our brains over this one for quite some time now, and, as I say, the cleverest number devils have tried every trick in the book. Sometimes we can work it out and sometimes we can't."

"Too bad," said Robert. "Sometimes is only halfway."

"What's worse, we can't even prove with any certainty that *no* perfect solution exists. That would be something at least. It would allow us to stop looking for one. Besides, proving that no proof exists is a proof in itself of sorts."

"Hm," said Robert. "It's comforting to know that number devils can fall on their faces. I thought you could conjure your way in and out of everything."

"It just looks that way. Many's the time I've been stuck in the middle of the stream. There are times I feel fortunate to get out with dry feet. Heaven knows I don't want to put myself up there with the greats, but even they—and you'll eventually come to know some of them—even they have their problems. Which only means that mathematics will never be over and done with. And a good thing too: there will always be plenty to keep us busy. So good-bye for now, my boy. First thing in the morning I'll be looking into the simplex algorithm for polytopic surfaces."

"The what?"

"The best way to unravel a foul-up. And for that I need a good night's sleep. I'm off to bed. Good night."

And with that, the number devil disappeared, leaving Robert to rock peacefully on the swinging bench.

I wonder what polytopic surfaces are, Robert thought, but then decided it didn't matter. All that mattered was that he didn't have to worry about Mr. Bockel anymore: whenever Mr. Bockel got on his back, the number devil would come to his rescue.

The night was mild and Robert gave himself up to the pleasure of rocking and dreaming, rocking and thinking about nothing at all till the break of day.